ph260 Theoretical Physics 2 — workshop 6

1. Fourier transforms.
Determine the Fourier transforms of the following functions:

a. $f(x) = \{-1 \quad (-\pi < x < 0); \qquad +1 \quad (0 < x < \pi); \qquad 0 \quad (\text{otherwise})\}$.

$g(q) = -\frac{1}{2\pi}\int_{-\pi}^{0} \mathrm{e}^{-jqx}\mathrm{d}x + \frac{1}{2\pi}\int_0^{\pi} \mathrm{e}^{-jqx}\mathrm{d}x = \frac{j}{2\pi q}\left(\mathrm{e}^{-jq\pi} + \mathrm{e}^{jq\pi} - 2\right) = \frac{j}{q\pi}\left(\cos q\pi - 1\right)$

b. $f(t) = \mathrm{e}^{-kt} g(t)$, where $g(t) = \{1 \quad (0 \le t \le b); \qquad 0 \quad (\text{otherwise})\}$.

$F(\omega) = \frac{1}{2\pi}\int 0b \mathrm{e}^{-kt}\mathrm{e}^{-j\omega t}\mathrm{d}t = \frac{1}{2\pi}\left[-\frac{\mathrm{e}^{-(k+j\omega)t}}{k+j\omega}\right]_0^b$

Now, note that in general the exponential will not have decayed to a value that is experimentally indistinguishable from zero. Thus,

$F(\omega) = \frac{1}{2\pi}\left(-\frac{\mathrm{e}^{-(k+j\omega)b}}{k+j\omega} + \frac{1}{k+j\omega}\right) = \frac{1}{2\pi}\frac{k-j\omega}{k^2+\omega^2}\left(1 - \mathrm{e}^{-kb}\mathrm{e}^{-jb\omega}\right)$

It is important to distinguish the difference between the first (real) exponential, which is a decaying function, and the second (complex) one, which is a phase factor, i.e. it represents $\cos b\omega + j\sin b\omega$. This causes the spectrum to have both cosine and sine components in both its real and imaginary parts (because the $j\omega$ in the prefactor combines with the $j\sin b\omega$ of the phase factor to form a real contribution).

2. Calculating a spectrum.
A complex time-domain signal is measured in an experiment. After an initial fast decay with time constant k_1, it decays exponentially with time constant k_2 and has two cosine waves superimposed whose periods are $1/c_1$ and $1/c_2$. For experimental reasons, only the time range from t_1 to t_2 can be observed. Find the function $f(t)$ that describes this behaviour and the corresponding spectrum $F(\omega)$ (You will get four terms for each combination of $k_{1,2}$ and $c_{1,2}$ – they all have the same form; so exploit the symmetry to keep the equations simple.). If you can get hold of a computer with mathematical software on it, plot the real and imaginary parts of both $f(t)$ and $F(\omega)$ and see how changes to the parameters $k_1, k_2, c_1, c_2, t_1, t_2$ affect the spectrum. In particular, investigate the following:

The time-domain signal is $f(t) = \left(\mathrm{e}^{-k_1 t} + \mathrm{e}^{-k_2 t}\right)\left(\mathrm{e}^{jc_1 t} + \mathrm{e}^{jc_2 t}\right)$. *The spectrum is*

$F(\omega) = \frac{1}{2\pi}\int_{t_1}^{t_2}\left(\mathrm{e}^{-k_1 t} + \mathrm{e}^{-k_2 t}\right)\left(\mathrm{e}^{jc_1 t} + \mathrm{e}^{jc_2 t}\right)\mathrm{e}^{-j\omega t}\mathrm{d}t$

$= \frac{1}{2\pi}\sum_{kc}\int_{t_1}^{t_2}\mathrm{e}^{-(k+j(\omega-c))t}\mathrm{d}t$, *where $\sum_{kc}$ is the sum of the four terms for each combination of $k_{1,2}$ and $c_{1,2}$.*

$= -\frac{1}{2\pi}\sum_{kc}\left[\frac{\mathrm{e}^{-(k+j(\omega-c))t}}{k+j(\omega-c)}\right]_{t_1}^{t_2} = -\frac{1}{2\pi}\sum_{kc}\left(\frac{k-j(\omega-c)}{k^2+(\omega-c)^2}\left(\mathrm{e}^{-(k+j(\omega-c))t_2} - \mathrm{e}^{-(k+j(\omega-c))t_1}\right)\right)$.

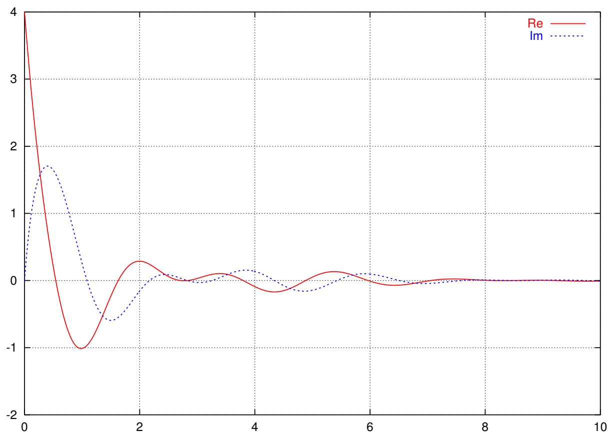
In order to separate the real and imaginary parts of the spectrum from each other the phase factor needs to be split up in its real and imaginary components:

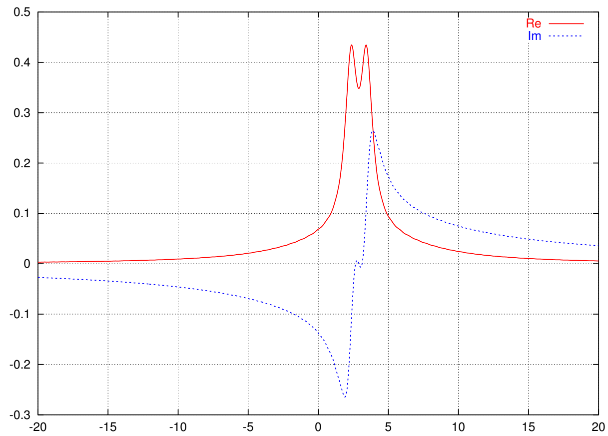
$= -\frac{1}{2\pi}\sum_{kc}\frac{k-j(\omega-c)}{k^2+(\omega-c)^2}\left(\mathrm{e}^{-kt_2}\left(\cos\left((\omega-c)t_2\right) + j\sin\left((\omega-c)t_2\right)\right) - \mathrm{e}^{-kt_1}\left(\cos\left((\omega-c)t_1\right) + j\sin\left((\omega-c)t_1\right)\right)\right)$.

By splitting the real parts from the imaginary parts, we have the real spectrum

$\mathcal{R}(F(\omega)) = -\frac{1}{2\pi}\sum_{kc}\left(\frac{k}{k^2+(\omega-c)^2}\left(\mathrm{e}^{-kt_2}\cos\left((\omega-c)t_2\right) - \mathrm{e}^{-kt_1}\cos\left((\omega-c)t_1\right)\right)\right.$
$\left. + \frac{\omega-c}{k^2+(\omega-c)^2}\left(\mathrm{e}^{-kt_2}\sin\left((\omega-c)t_2\right) - \mathrm{e}^{-kt_1}\sin\left((\omega-c)t_1\right)\right)\right)$, *and*

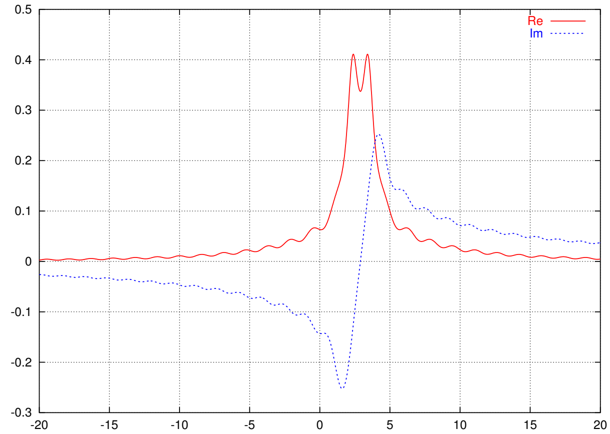
$\mathcal{I}(F(\omega)) = \frac{1}{2\pi}\sum_{kc}\left(\frac{\omega-c}{k^2+(\omega-c)^2}\left(\mathrm{e}^{-kt_2}\cos\left((\omega-c)t_2\right) - \mathrm{e}^{-kt_1}\cos\left((\omega-c)t_1\right)\right)\right.$
$\left. + \frac{k}{k^2+(\omega-c)^2}\left(\mathrm{e}^{-kt_2}\sin\left((\omega-c)t_2\right) - \mathrm{e}^{-kt_1}\sin\left((\omega-c)t_1\right)\right)\right)$.





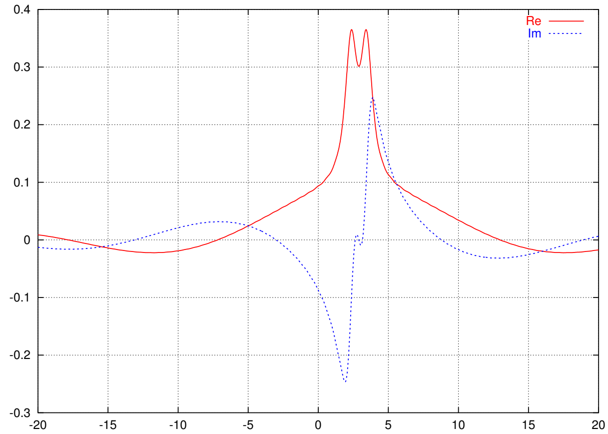
a. What is the effect of cutting off the decay before it is close to zero (*i.e.* too small t_2)?

In the image below, the cutoff point is set to $t_2 = 4$*, where there is still a lot of signal present in the time domain. As a consequence, wiggles appear in both real and imaginary part.*



b. What is the effect of starting the time domain at $t_1 > 0$?

In the image below, the time-domain signal is recorded from $t_1 = 0.3$*. As a consequence, the baselines of both real and imaginary part are distorted by a wave with a long period.*



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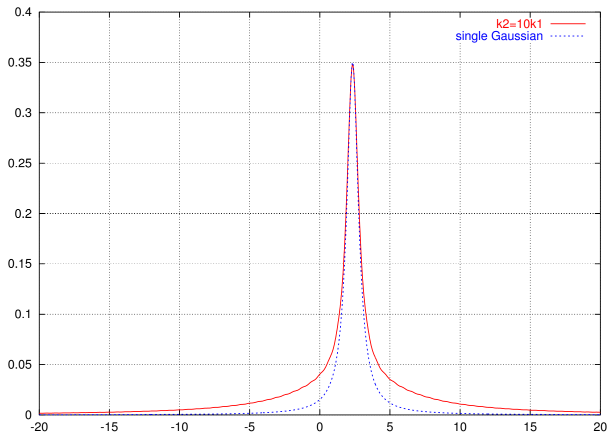
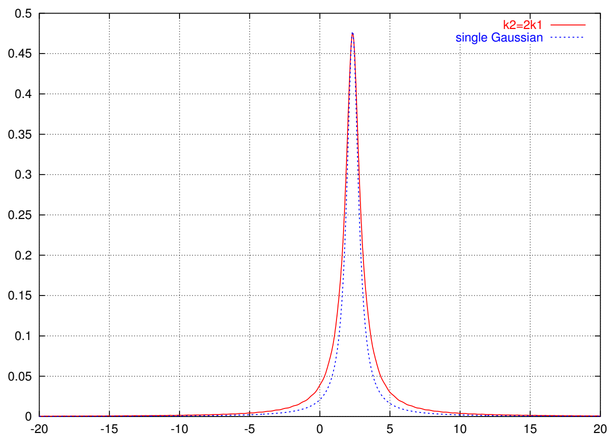
c. What is the minimum ratio $\frac{k_1}{k_2}$ for which the spectral contributions from the two decays remain discernible?

The two images below show spectra (with one oscillation only) of double-exponential signals where the fast decay is ten times (top) and two times (bottom) as fast as the slow one. The narrower curves are single Gaussian lines to show the contribution of the fast (spectrally broad) component.





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d. What is the relative minimum frequency difference $\delta c = \frac{c_1 - c_2}{c_1}$ of the two waves to ensure they come out as distinct peak maxima in the spectrum?

The two figures below consist of spectra containing two frequencies and a single decay. On the top, $\delta c = 1$ *produces a clear separation of the maxima; on the bottom,* $\delta c = 0.25$ *results in a single peak, but note its broadening near the top.*