ph260 Theoretical Physics 2 — workshop 5

1. Wave equation.
In the lecture, we have worked out the equation of motion of a *plucked* string, *i.e.* a vibrating string of length l whose motion begins with the string initially at rest but in a non-equilibrium position. Its boundary conditions were $y(x,0) = \begin{cases} \frac{2hx}{l} \; (0 \le x \le \frac{l}{2}) \\ \frac{2h(l-x)}{l} \; (\frac{l}{2} \le x \le l) \end{cases}$ and $\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = 0$.

If the string is *hit* rather than plucked, *e.g.* in a piano, the boundary conditions are opposite: The string is in relaxed position at the beginning, but due to the impact of the hammer, it is at maximum velocity. This initial velocity is proportional to the distance from the ends of the string. Cast the boundary conditions in mathematical formalism, choose the correct general solution of the wave equation, apply the boundary conditions, and find the series solution of $y(x,t)$ of the piano string.

BCs: $y(x,0) = 0$ *and* $\left.\frac{\partial y(x,t)}{\partial t}\right|_{t=0} = v(x) = \begin{cases} \frac{2hx}{l} \; (0 \le x \le \frac{l}{2}) \\ \frac{2h(l-x)}{l} \; (\frac{l}{2} \le x \le l) \end{cases}$ *(where h is in units of velocity this time).*
General solution $y(x,t) = \sin\frac{n\pi x}{l} \sin\frac{n\pi vt}{l}$ *applies here since* $\cos 0 = 1$.
Expand in series: $y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \sin\frac{n\pi vt}{l}$.
Apply BC (2): $\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{l} \sin\frac{n\pi x}{l} \cos\frac{n\pi vt}{l}$. *Hence, at* $t=0$, $\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} b_n \frac{n\pi v}{l} \sin\frac{n\pi x}{l} = v(x)$.
With $d_n = b_n \frac{n\pi v}{l}$*, this gives the Fourier sine series again (exactly as for the plucked string).*
The solution was $d_n = \frac{8h}{n^2\pi^2} \sin\frac{n\pi}{2}$. *Hence,* $b_n = \frac{d_n l}{n\pi v} = \frac{8hl}{n^3\pi^3 v} \sin\frac{n\pi}{2}$.
Substitute into $y(x,t)$ *(not* $\frac{\partial y}{\partial t}$ *!):* $y(x,t = \sum_{\mathrm{odd}\, n}) = \frac{8hl(-1)^{\frac{n-1}{2}}}{n^3\pi^3 v} \sin\frac{n\pi x}{l} \sin\frac{n\pi vt}{l}$.

2. Cartesian drum.
Drums are usually circular – for mathematically sound reasons as we'll see. Write down the wave equation $z(x,y,t) = \ldots$ for a "Cartesian drum", *i.e.* a drum with a membrane fixed to a rectangular frame of size $a \times b$. Separate the equation into a spatial and a temporal part and find the general solutions. Work out the boundary conditions of the problem (assuming that the drum is hit with a drumstick to start the oscillation), decide which of the general solutions can be discarded because they conflict with the BCs, and write down a linear combination of the remaining ones as a preliminary solution of this wave equation. Will the Cartesian drum's sound be harmonic?

Equation: $z(x,y,t) = \nabla z = \frac{1}{v^2} \frac{\partial^2 z}{\partial t^2}$.
SoV ($z(x,y,t) = A(x,y)T(t)$)*:* $\frac{1}{A} \nabla A = \frac{1}{v^2 T} \frac{\mathrm{d}^2 T}{\mathrm{d}t^2} = -k^2$.
temporal eq. – 2nd order ODE with const. coeff: $T(t) = \cos kvt$ or $\sin kvt$.
spatial eq. – SoV ($A(x,y) = X(x)Y(y)$)*:* $\frac{1}{X} \frac{\mathrm{d}^2 X}{\mathrm{d}x^2} + k^2 = -\frac{1}{Y} \frac{\mathrm{d}^2 Y}{\mathrm{d}y^2} = -l^2$.
Using $-l^2$ *rather than* $+l^2$ *as separation constant determines that the solution oscillates along* x *but is damped along* y*. However, the problem is symmetric w.r.t. swapping* x *and* y*; there is a damped oscillation along both. Thus, we need a linear combination of both solutions for both variables:*
$X(x) = c_1 \sin\sqrt{k^2 + l^2}\,x + c_2 \exp \pm lx$ *and the same for* $Y(y)$*. The* $\cos$ *solutions have been discarded since they would not be zero where the membrane is attached to the frame.*
Since the membrane is finite both in x *and* y *direction, we can't discard the increasing exponential solution. Instead, the decaying and increasing exponentials are combined into a* sinh *function as in last week's question 2:*
$X(x) = c_1 \sin\sqrt{k^2 + l^2}\,x + c_2 \sinh\left(l(a-x)\right)$ *and similar for* $Y(y)$.
Finally, we can apply the periodic conditions $\sqrt{k^2 + l^2} = \frac{n\pi}{a}$ *and* $\sqrt{k^2 + l^2} = \frac{m\pi}{b}$.
From here on, you could apply the remaining BCs to get Fourier series in both x *and* y *direction, resulting in a double sum over* n *and* m.
The Cartesian drum isn't harmonic because the sinh *term distorts the frequency (even for* $a = b$*). A proper drum has a circular membrane, and its wave equation is easy to solve in polar coordinates. Since a circular membrane oscillates along the radial direction only, the resulting sound is harmonic, i.e. contains only harmonics of the* $n = 1$ *solution.*

Acknowledgement.
Example 1 is stolen from *ML Boas; Mathematical Methods in the Physical Sciences, John Wiley, New York (USA)* ^{2}*1983.*

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