ph260 Theoretical Physics 2 — workshop 4

1. Laplace equation.
Determine the temperature profile in a long thin plate 10 length units wide and kept at zero temperature at the long sides and the far short edge. It is heated at one corner of the remaining short edge; the resulting temperature profile at the edge is approximately proportional to x. Let the long side be orientated along the y axis.
The BCs are: $T(0,y) = T(10,y) = T(x,\infty) = 0$ and $T(x,0) = x$ (strictly ax, but since we have arbitrary units we can as well set the constant of proportionality to one).
Of the eight general solutions of the Laplace equation, we can eliminate:
— the ones that oscillate along y because of $T(x,\infty) = 0$,
— the ones that increase along y because of $T(x,\infty) = 0$,
— $T(x,y) = \mathrm{e}^{-ky}\cos kx$ because of $T(0,y) = 0$.
The remaining one is $T(x,y) = \mathrm{e}^{-ky}\sin kx$.
Applying $T(10,y) = 0$ requires that $\sin 10k = 0$, i.e. $k = \frac{n\pi}{10}$ $(n = 0,1,2,\ldots)$.
The solution is a linear combination of the solutions with all possible values of k:
$T(x,y) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{10}\mathrm{e}^{-\frac{n\pi y}{10}}$.
The Fourier coefficients are found by: $b_n = \frac{2}{10}\int_0^{10} x\sin\frac{n\pi x}{10}\mathrm{d}x$,
which results in $b_n = \pm\frac{20}{n\pi}$ for (odd/even n).
Thus the solution is $T(x,y) = \sum_{n=1}^{\infty}\frac{20(-1)^{n+1}}{n\pi}\sin\frac{n\pi x}{10}\mathrm{e}^{-\frac{n\pi y}{10}}$ (the $n = 0$ term is zero anyway).

2. Laplace equation in a finite plate.
Same as above, but chop the long plate off at 30 length units and keep it at zero temperature there.
$T(x,\infty) = 0$ is now replaced by $T(x,30) = 0$.
Keep the increasing exponential in the solution and ensure that the two terms cancel at $y = 30$:
$T(x,y) = \left(\frac{1}{2}\mathrm{e}^{k(30-y)} - \frac{1}{2}\mathrm{e}^{-k(30-y)}\right)\sin kx = \sinh(k(30-y))\sin kx$.
Applying $T(10,y) = 0$ requires that $\sin 10k = 0$, i.e. $k = \frac{n\pi}{10}$ $(n = 0,1,2,\ldots)$.
The solution is a linear combination of the solutions with all possible values of k:
$T(x,y) = \sum_{n=0}^{\infty} b_n \sinh(k(30-y))\sin kx$.
With $d_n = b_n \sinh 3\pi n$, this turns into the familiar sine series; the d_n are the b_n of the previous example.
Solution: $T(x,y) = \sum_{n=1}^{\infty} b_n \frac{20(-1)^{n+1}}{n\pi\sinh 3\pi n}\sin\frac{n\pi x}{10}\sinh\frac{(30-y)n\pi}{10}$.

3. Diffusion equation.
A bar 20 length units long with insulated sides is initially at 100 temperature units, except for the ends, which are kept at zero temperature throughout the experiment. Find the temperature distribution in the bar at time t.
BCs: $T(0,t) = T(20,t) = 0$ and $T(x,0) = 100$.
In the 2D case, the diffusion equation is solved generally by:
$T(x,y,t) = \mathrm{e}^{\pm ly-\alpha^2k^2t}\sin\sqrt{k^2+l^2}x$ (or the corresponding $\cos$ term or either with x and y swapped).
In the 1D case here, the y-term and the separation constant k aren't necessary, hence:
$T(x,t) = \mathrm{e}^{-\alpha^2k^2t}\sin kx$ (or the corresponding $\cos$ term).
The $\sin$ version is needed here because of $T(0,t) = 0$.
Applying $T(20,t) = 0$ results in $k = \frac{n\pi}{20}$ $(n = 0,1,2,\ldots)$ as usual.
The sum of linear combinations of the possible solutions is $T(x,t) = \sum_{n=0}^{\infty} b_n\mathrm{e}^{-\frac{\alpha^2n^2\pi^2t}{400}}\sin\frac{n\pi x}{20}$,
where $b_n = \frac{400}{n\pi}$ for odd n only (and 0 otherwise).
Solution: $T(x,t) = \sum_{n=1}^{\infty}\frac{400(-1)^{n+1}}{n\pi}\mathrm{e}^{-\frac{\alpha^2n^2\pi^2t}{400}}\sin\frac{n\pi x}{20}$.

4. Even and odd expansions.
In the lecture, we have extended a non-periodic function, which is defined in a limited range only, to an odd periodic function (*odd* meaning $f(-x) = -f(x)$), so that we were able to expand it in a sine series. The function used in the example was $f(x) = \begin{cases} 1 & (0<x<\pi) \\ 0 & (\text{otherwise})\end{cases}$. We could just as well have extended $f(x)$ to an *even* function ($f(-x) = f(x)$) and expanded it into a cosine series. Try it, find the coefficients, and show that the result is the same within the range in which $f(x)$ is defined by plotting the first terms.
The simplest fitting even function is alternating between ± 1 with a half period of $l = 2\pi$, centred on $x = 0$. Note that this is twice the period needed for the sine series.
The cosine expansion is $f(x) = \sum_{n=0}^{\infty} a_n\cos\frac{n\pi x}{l}$. Note that here the $n = 0$ term is usually necessary.
The Fourier coefficients are $a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\mathrm{d}x$.

In this example, $a_n = 0$ and $a_n = \pm\frac{2}{n\pi}$ for even and odd n, respectively (with $n = 1, 5, 9, \ldots$ producing the positive solutions and $n = 3, 7, 11, \ldots$ the negative ones).

Acknowledgement.
Examples 1-3 are stolen or adapted from *ML Boas; Mathematical Methods in the Physical Sciences, John Wiley, New York (USA) ^{2}1983.*

rw/031023