ph260 Theoretical Physics 2 — workshop 2 — solutions

1. Classifying differential equations.
Determine the order of the following differential equations, state whether they are linear or non-linear, homogeneous or non-homogeneous and ordinary or partial. How many boundary conditions will you need to find a specific solution for each? Find examples of the type of differential equation described at the bottom of the table and state how many boundary conditions you need for your example. (a,b,c,d are just non-zero constants.)

	order	linear?	ordinary?	homo-geneous?	number of BCs
$\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} + 3y = 13$	2	+	−	−	3
$\frac{\mathrm{d}y}{\mathrm{d}x} + ay = 0$	1	+	+	+	1
$\frac{\mathrm{d}y}{\mathrm{d}x} - a\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -y$	1	−	+	+	1
$\frac{\partial z}{\partial x} + 5z + 3y^2 = a$	1	−	−	−	1
$\frac{\partial^3 z}{\partial x^2 \partial y} + \frac{b}{z} = 0$	3	+	−	+	3
$a\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x}$	1	+	−	+	2
$\frac{\mathrm{d}y}{\mathrm{d}x} - a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -y$	2	+	+	+	2
$\frac{\mathrm{d}y}{\mathrm{d}x} + ay^3 + by^2 + cy + d = 0$	1	−	+	−	1
$e.g. \frac{\partial^3 z(x,y)}{\partial x^3} + az = 0$	3	+	−	+	3
$e.g. \frac{\partial z}{\partial x} + ay + bz = c$	1	+	−	−	1
$e.g. \sin\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) + ay = 0$	2	−	+	+	1

2. Picking solution strategies for ODEs.
Decide whether you can solve the following ODEs by separation, by using the general approach for linear ODEs, by using Bernoulli's equation or by applying the homogeneous-equation approach, or whether it is one of the stubborn cases that need reading up in a maths book... Unless the latter is the case, solve it, then apply boundary conditions where supplied. If you can't solve the equation, say why each of the techniques fail.

a. $xy' - xy = y \quad ; \quad y(1) = 1$
Solve by separation: $\frac{1}{y}\mathrm{d}y = \frac{x+1}{x}\mathrm{d}x$.
Result: $y(x) = x\mathrm{e}^{x-1}$.

b. $y' + y\cos x = \sin 2x$
Not separable: 3 terms, one of which will always contain both x and y.
Solve by general formula for fully linear ODE: $y' + \cos(x)y = sin(2x)$.
Applying the formula yields: $y(x) = \mathrm{e}^{-\sin x}\left(\int \sin(2x)\mathrm{e}^{\sin x}\mathrm{d}x + c\right)$.
I guess that this can be solved by substitution, but I haven't tried.

c. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy^2+x}{x^2y-y}$
Solve by separation: $\frac{x}{x^2-1}\mathrm{d}x = \frac{y}{2y^2+1}\mathrm{d}y$.
Each side then takes a product rule ($\int fg\mathrm{d}x = f\int g\mathrm{d}x - \int \frac{\mathrm{d}f}{\mathrm{d}x}\int g\mathrm{d}^2x$) to eliminate the x or y in the denominator. The remaining integrals are of the form $\frac{1}{ax^2+bx+c}$ and can be looked up.

d. $3xy^2y' + 3y^3 = 1 \quad ; \quad y(0) = -8$
Not separable: 3 terms, one of which will always contain both x and y.
Not fully linear: y^3 term prevents usage of general formula.
Solve by Bernoulli approach: $y' + \frac{1}{x}y = \frac{1}{3x}y^{-2}$.
Note that $\mathrm{e}^{-a\ln(x)+c} = b\mathrm{e}^{-a\ln(x)} = b\left(\mathrm{e}^{\ln(x)}\right)^{-a} = bx^{-a}$ *(where $b = e^c$).*
Result prior to applying boundary condition: $y = \left(\frac{1}{3} - cx^{-3}\right)^{\frac{1}{3}}$.
I'm afraid I got the boundary condition wrong - it isn't consistent with the general solution.

e. $x^2y' + 3xy = 1 \quad ; \quad y(3) = 0$
Not separable: 3 terms, one of which will always contain both x and y.
Solve by general formula for fully linear ODE: $y' + \frac{3}{x}y = \frac{1}{x^2}$.
Result: $y(x) = \frac{1}{2x} - \frac{9}{2x^3}$.

f. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y - x\tan y} \quad ; \quad y(0) = \pi$
This one is better solved upside down - solve for x(y) rather than y(x): $\frac{\mathrm{d}x}{\mathrm{d}y} = \cos(y) - x\tan(y)$.

Not separable: 3 terms, one of which will always contain both y and x.
Solve by general formula for fully linear ODE: $x' + \tan(y)x = \cos y$.
Result: $x(y) = (y - \pi)\cos y$. If you are pedantic about it, you could rearrange this to yield $y(x) = \ldots$

g. $$xy\mathrm{d}x + (y^2 - x^2)\mathrm{d}y = 0$$
Not separable: 3 terms, one of which will always contain both x and y.
Not fully linear: y^2 term prevents usage of general formula.
Not Bernoulli-type: Prefactor $p(x)$ cannot be made function of x only.
Homogeneous equation - substitute $v = \frac{y}{x}$, then separate: $\frac{1}{x}\mathrm{d}x = \frac{1-v^2}{v}\mathrm{d}v$.
Result after resubstitution: $y = x^2\mathrm{e}^{\frac{y^2}{2x^2}}$ (note that this is a circular formula containing y on either side).

h. $$\cos x \cos y\mathrm{d}x - \sin x \sin y\mathrm{d}y = 0 \quad ; \qquad y(\tfrac{\pi}{2}) = \pi$$
Solve by separation: $\frac{\cos x}{\sin x}\mathrm{d}x = \frac{\sin y}{\cos y}\mathrm{d}y$.
Result: $\sin x \cos y = -1$. Again, you could rearrange this to get it in $y(x) = \ldots$ form.

Don't forget that you can (and should!) always check your result by substituting it into the original differential equation!

3. Translating physics into maths.

Take the following physical problems and write them down as a formula. Then replace the variables in the formula by those you are used to from the maths toolbox. Say which approach will solve the equation and solve it.

a. Derive a formula for the growth of an ice layer on a lake in cold weather. To keep the problem simple, assume the temperature of the liquid is a constant T_l=283 K, the air above a constant T_g=263 K, and the ice grows in a layer of uniform thickness $x(t)$ as time t progresses. The rate of formation of ice is proportional to the rate at which heat is transferred from the liquid to the air above. Start at the moment just before ice formation begins.
Heat transfer is proportional to the temperature gradient $\frac{T_l - T_g}{x}$.
This leads to the ODE $\frac{\mathrm{d}x}{\mathrm{d}t} = \kappa\frac{T_l - T_g}{x}$, where κ is the thermal conductivity of ice.
Translate into maths toolbox notation if you want ($x \to y$, $t \to x$, $\kappa(T_l - T_g) = a$): $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{a}{x}$.
Solve by separation (in physics notation): $x\mathrm{d}x = \kappa(T_l - T_g)\mathrm{d}t$.
Result: $x(t) = \sqrt{2\kappa(T_l - T_g)t}$ using the boundary condition $x(0) = 0$.

b. The decay sequence of Uranium is ${}^{238}\mathrm{U} \overset{\alpha}{\to} {}^{234}\mathrm{Th} \overset{\beta}{\to} {}^{234}\mathrm{Pa} \overset{\beta}{\to} {}^{234}\mathrm{U} \overset{\alpha}{\to} {}^{230}\mathrm{Th} \overset{\alpha}{\to} {}^{226}\mathrm{Ra} \overset{\alpha}{\to} {}^{Rn}222 \overset{\alpha}{\to} {}^{218}\mathrm{Po} \overset{\alpha}{\to} {}^{214}\mathrm{Pb} \overset{\beta}{\to} {}^{214}\mathrm{Bi} \overset{\beta}{\to} {}^{214}\mathrm{Po} \overset{\alpha}{\to} {}^{210}\mathrm{Pb} \overset{\beta}{\to} {}^{210}\mathrm{Bi} \overset{\beta}{\to} {}^{210}\mathrm{Po} \overset{\alpha}{\to} {}^{206}\mathrm{Pb}$ (stable). Work out the amount N_{15} of stable ^{206}Pb as a function of time if you start with a slab of ^{238}U containing N_0 atoms. The half lives of the isotopes are denoted by λ_i for isotope i in the chain, starting from i=1 for ^{238}U.

Hint: First assume that there is only one step rather than a chain of decays. Then add the second step. Then work out by analogy the formula for the i-th step. You can then either explicitly work yourself through the whole sequence or establish a recursive formula.

If there were only one step, $\frac{\mathrm{d}N_1}{\mathrm{d}t} = -\lambda_1 N_1$, which can be solved by separation.
Result: $N_1 = N_0\mathrm{e}^{-\lambda_1 t}$.
In the second step, the rate of change of the ^{234}Th population is a balance of decaying ^{234}Th nuclei and newly formed ^{234}Th nuclei formed by the decay of ^{238}U: $\frac{\mathrm{d}N_2}{\mathrm{d}t} = \lambda_1 N_1 - \lambda_2 N_2$.
Not separable: 3 terms, one of which will always contain both t and N_2.
Solve by general formula for fully linear ODE: $\frac{\mathrm{d}N_2}{\mathrm{d}t} + \lambda_2 N_2 = \lambda_1 N_1$. (Note that N_2 is the dependent variable while N_1 is a function of the independent variable t only.)
Result: $N_2 = \frac{\lambda_1 N_1}{\lambda_2 - \lambda_1}\left(\mathrm{e}^{-\lambda_1 t} - \mathrm{e}^{\lambda_2 t}\right)$ using the boundary condition $N_2(0) = 0$ (i.e. pure ^{238}U at first).
Similarly, for the i-th step, we have $\frac{\mathrm{d}N_i}{\mathrm{d}t} = \lambda_{i-1}N_{i-1} - \lambda_i N_i$ and hence $N_i = \frac{\lambda_{i-1}N_{i-1}}{\lambda_i - \lambda_{i-1}}\left(\mathrm{e}^{-\lambda_{i-1}t} - \mathrm{e}^{\lambda_i t}\right)$.
Because of the recursion in this formula, the amount of isotope i at any given time is
$N_i = \frac{\lambda_{i-1}}{\lambda_i - \lambda_{i-1}}\left(\mathrm{e}^{-\lambda_{i-1}t} - \mathrm{e}^{\lambda_i t}\right) \times \frac{\lambda_{i-2}}{\lambda_{i-1} - \lambda_{i-2}}\left(\mathrm{e}^{-\lambda_{i-2}t} - \mathrm{e}^{\lambda_{i-1}t}\right) \times \cdots \times \frac{\lambda_1}{\lambda_2 - \lambda_1}\left(\mathrm{e}^{-\lambda_1 t} - \mathrm{e}^{\lambda_2 t}\right) \times N_0\mathrm{e}^{-\lambda_1 t}$,
or, using the symbol $\Pi_{j=1}^{i-1}$ for the product from index $j = 1$ to $j = i - 1$,
$N_i = N_0\mathrm{e}^{-\lambda_1 t}\Pi_{j=1}^{i-1}\frac{\lambda_j}{\lambda_{j+1} - \lambda_j}\left(\mathrm{e}^{-\lambda_j t} - \mathrm{e}^{-\lambda_{j+1}t}\right)$.

After having found the solution algebraically, you may wish to plug in the numbers and think about implications for the deep-level storage of nuclear waste. Here are the half lives in years, days, or seconds.

$\lambda_1 = 4.468 \times 10^9$ a	$\lambda_4 = 244\,600$ a	$\lambda_7 = 3.825$ d	$\lambda_{10} = 1\,194$ s	$\lambda_{13} = 5.013$ d
$\lambda_2 = 24.10$ d	$\lambda_5 = 75\,400$ a	$\lambda_8 = 183$ s	$\lambda_{11} = 1.64 \times 10^{-4}$ s	$\lambda_{14} = 138.38$ d
$\lambda_3 = 70.2$ s	$\lambda_6 = 1\,600$ a	$\lambda_9 = 1\,608$ s	$\lambda_{12} = 22.3$ a	

This is one of the three naturally occurring decay sequences. Basically, the first step is so slow that all other steps can be regarded as "instantaneous" on its time scale. Most atoms in any given sample from the sequence *will be either* ^{238}U *or* ^{206}Pb. *However,* ^{238}U *is no use as fuel (partly because of its relative stability), and intermediate products do play an important role when predicting the behaviour of nuclear waste.*

4. Deriving the derivative.

When dealing with the Bernoulli equation approach, we've used the substitution $z = y^{1-n}$ and its derivative $\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$. Show that this derivative is in fact correct.

$\frac{\mathrm{d}z}{\mathrm{d}x} = \frac{\mathrm{d}z}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}y}\left(y^{1-n}\right)\frac{\mathrm{d}y}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$.

Note that although the first step looks just like a multiplication by $\frac{\mathrm{d}y}{\mathrm{d}y}$*, it isn't: in actual fact, the chain rule is applied to the function* $z(y(x))$ *here.*

Acknowledgement.

Most of these examples are stolen or adapted from *ML Boas; Mathematical Methods in the Physical Sciences, John Wiley, New York (USA)* ^{2}*1983.*

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