ph260 Theoretical Physics 2 — workshop 2

1. Classifying differential equations.
Determine the order of the following differential equations, state whether they are linear or non-linear, homogeneous or non-homogeneous and ordinary or partial. How many boundary conditions will you need to find a specific solution for each. Find examples of the type of differential equation described at the bottom of the table and state how many boundary conditions you need for your example. (a,b,c,d are just non-zero constants.)

	order	linear?	ordinary?	homo-geneous?	number of BCs
$\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} + 3y = 13$	2	+	−	−	3
$\frac{\mathrm{d}y}{\mathrm{d}x} + ay = 0$					
$\frac{\mathrm{d}y}{\mathrm{d}x} - a\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -y$					
$\frac{\partial z}{\partial x} + 5z + 3y^2 = a$					
$\frac{\partial^3 z}{\partial x^2 \partial y} + \frac{b}{z} = 0$					
$a\frac{\partial z}{\partial y} = x\frac{\partial z}{\partial x}$					
$\frac{\mathrm{d}y}{\mathrm{d}x} - a\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} = -y$					
$\frac{\mathrm{d}y}{\mathrm{d}x} + ay^3 + by^2 + cy + d = 0$					
	3	+	−	+	
	1	+	−	−	
	2	−	+	+	

2. Picking solution strategies for ODEs.
Decide whether you can solve the following ODEs by separation, by using the general approach for linear ODEs, by using Bernoulli's equation or by applying the homogeneous-equation approach, or whether it is one of the stubborn cases that need reading up in a maths book... Unless the latter is the case, solve it, then apply boundary conditions where supplied. If you can't solve the equation, say why each of the techniques fail.

a. $xy' - xy = y \quad ; \quad y(1) = 1$

b. $y' + y\cos x = \sin 2x$

c. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2xy^2+x}{x^2y-y}$

d. $3xy^2y' + 3y^3 = 1 \quad ; \quad y(0) = -8$

e. $x^2y' + 3xy = 1 \quad ; \quad y(3) = 0$

f. $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{\cos y - x\tan y} \quad ; \quad y(0) = \pi$

g. $xy\mathrm{d}x + (y^2 - x^2)\mathrm{d}y = 0$

h. $\cos x\cos y\mathrm{d}x - \sin x\sin y\mathrm{d}y = 0 \quad ; \quad y(\frac{\pi}{2}) = \pi$

3. Translating physics into maths.
Take the following physical problems and write them down as a formula. Then replace the variables in the formula by those you are used to from the maths toolbox. Say which approach will solve the equation and solve it.

a. Derive a formula for the growth of an ice layer on a lake in cold weather. To keep the problem simple, assume the temperature of the liquid is a constant T_l=283 K, the air above a constant T_g=263 K, and the ice grows in a layer of uniform thickness $x(t)$ as time t progresses. The rate of formation of ice is proportional to the rate at which heat is transferred from the liquid to the air above. Start at the moment just before ice formation begins.

b. The decay sequence of Uranium is $^{238}\mathrm{U} \xrightarrow{\alpha} {}^{234}\mathrm{Th} \xrightarrow{\beta} {}^{234}\mathrm{Pa} \xrightarrow{\beta} {}^{234}\mathrm{U} \xrightarrow{\alpha} {}^{230}\mathrm{Th} \xrightarrow{\alpha} {}^{226}\mathrm{Ra} \xrightarrow{\alpha} {}^{Rn}222 \xrightarrow{\alpha} {}^{218}\mathrm{Po} \xrightarrow{\alpha} {}^{214}\mathrm{Pb} \xrightarrow{\beta} {}^{214}\mathrm{Bi} \xrightarrow{\beta} {}^{214}\mathrm{Po} \xrightarrow{\alpha} {}^{210}\mathrm{Pb} \xrightarrow{\beta} {}^{210}\mathrm{Bi} \xrightarrow{\beta} {}^{210}\mathrm{Po} \xrightarrow{\alpha} {}^{206}\mathrm{Pb}\,(\text{stable})$. Work out the amount N_{15} of stable ^{206}Pb as a function of time if you start with a slab of ^{238}U containing N_0 atoms. The half lives of the isotopes are denoted by λ_i for isotope i in the chain, starting from i=1 for ^{238}U.

Hint: First assume that there is only one step rather than a chain of decays. Then add the second step. Then work out by analogy the formula for the i-th step. You can then either explicitly work yourself through the whole sequence or establish a recursive formula.

After having found the solution algebraically, you may wish to plug in the numbers and think about implications for the deep-level storage of nuclear waste. Here are the half lives in years, days, or seconds.

$\lambda_1 = 4.468 \times 10^9\,a$	$\lambda_4 = 244\,600\,a$	$\lambda_7 = 3.825\,d$	$\lambda_{10} = 1\,194\,s$	$\lambda_{13} = 5.013\,d$
$\lambda_2 = 24.10\,d$	$\lambda_5 = 75\,400\,a$	$\lambda_8 = 183\,s$	$\lambda_{11} = 1.64 \times 10^{-4}\,s$	$\lambda_{14} = 138.38\,d$
$\lambda_3 = 70.2\,s$	$\lambda_6 = 1\,600\,a$	$\lambda_9 = 1\,608\,s$	$\lambda_{12} = 22.3\,a$	

4. Deriving the derivative.

When dealing with the Bernoulli equation approach, we've used the substitution $z = y^{1-n}$ and its derivative $\frac{\mathrm{d}z}{\mathrm{d}x} = (1-n)y^{-n}\frac{\mathrm{d}y}{\mathrm{d}x}$. Show that this derivative is in fact correct.

Acknowledgement.

Most of these examples are stolen or adapted from *ML Boas; Mathematical Methods in the Physical Sciences, John Wiley, New York (USA)* ^{2}*1983.*

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